MDOF SYSTEMS WITH DAMPING

MDOF Systems with hysteretic damping- general case

Free vibration solution:

$$[M]\{\ddot{x}\}+([K]+i[D])\{x\}=\{0\}$$

Assume a solution in the form of:

$$\{x\}=\{X\}e^{i\lambda t}$$

Here λ can be a complex number. The solution here is like the undamped case. However, both eigenvalues and Eigenvector matrices are complex.

The eigensolution has the orthogonal properties as:

$$[\Psi]^T[M][\Psi]=[m_r]; \qquad [\Psi]^T[K+iD][\Psi]=[k_r]$$

The modal mass and stiffness parameters are complex.

MDOF Systems with hysteretic damping- general case

Again, the following relation is valid:

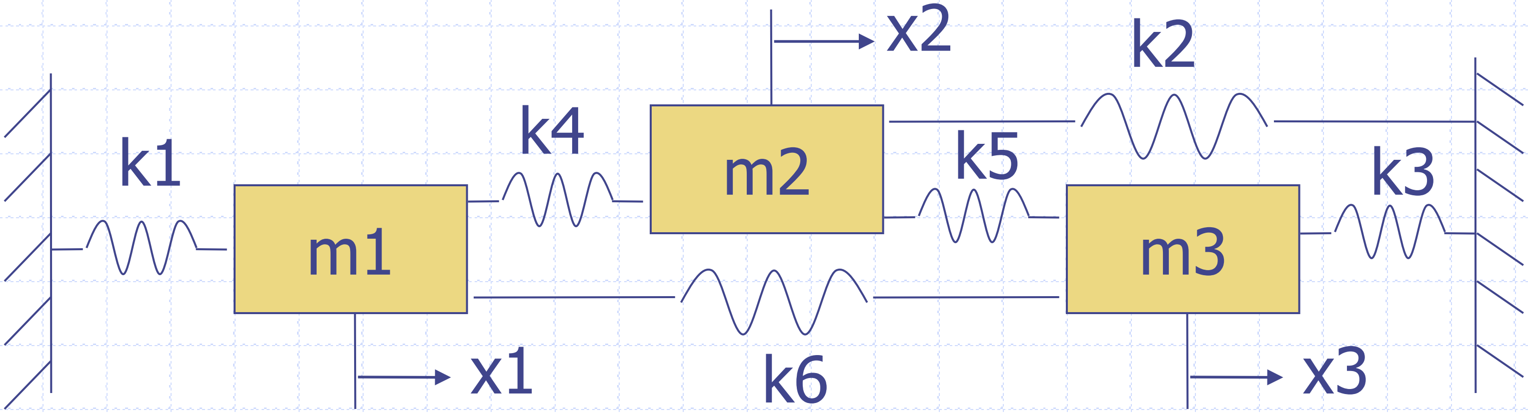
$$\lambda_r^2 = \frac{k_r}{m_r} = \omega_r^2 (1 + i\eta_r)$$

A set of mass-normalized eigenvectors can be defined as:

$$\{\phi\}_r = (m_r)^{-1/2} \{\psi\}_r$$

What is the interpretation of complex mode shapes?
The phase angle in undamped is either 0 or 180.
Here the phase angle may take any value.

Numerical Example with structural damping



m1=0.5 Kg
m2=1.0 Kg
m3=1.5 Kg
k1=k2=k3=k4=k5=k6=1000 N/m

Undamped

$$[M]=\begin{bmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \qquad [K]=\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

Using command [V,D]=eig(k,M) in MATLAB

$$[\omega_r^2]=\begin{bmatrix} 950 & 0 & 0 \\ 0 & 3352 & 0 \\ 0 & 0 & 6698 \end{bmatrix} \quad [\Phi]=\begin{bmatrix} .464 & -.218 & -1.318 \\ .536 & -.782 & .318 \\ .635 & .493 & .142 \end{bmatrix}$$

Proportional Structural Damping

Assume proportional structural damping as:

$$d_j = 0.05 k_j, \qquad j = 1, \ldots, 6$$

$$[\lambda_r^2] = \begin{bmatrix} 950(1+.05i) & 0 & 0 \\ 0 & 3352(1+.05i) & 0 \\ 0 & 0 & 6698(1+.05i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} .464(0^\circ) & .218(180^\circ) & 1.318(180^\circ) \\ .536(0^\circ) & .782(180^\circ) & .318(0^\circ) \\ .635(0^\circ) & .493(0^\circ) & .142(0^\circ) \end{bmatrix}$$

Non-Proportional Structural Damping

Assume non-proportional structural damping as:

$$d_1 = 0.3k_1$$

$$d_j = 0, \qquad j = 2,\ldots,6$$

$$[\lambda_r^2] = \begin{bmatrix} 957(1+.067i) & 0 & 0 \\ 0 & 3354(1+.042i) & 0 \\ 0 & 0 & 6690(1+.078i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} .463(-5.5^\circ) & .217(173^\circ) & 1.321(181^\circ) \\ .537(0^\circ) & .784(181^\circ) & .316(-6.7^\circ) \\ .636(0^\circ) & .492(-1.3^\circ) & .142(-3.1^\circ) \end{bmatrix}$$

Non-Proportional Structural Damping

- Each mode has a different damping factor.
- All eigenvectors arguments for undamped and proportional damp cases are either 0 or 180.
- All eigenvectors arguments for non-proportional case are within 10 degree of 0 or 180 (the modes are almost real).

Exercise: Repeat the problem with

m1=1Kg, m2=0.95 Kg, m3=1.05 Kg

k1=k2=k3=k4=k5=k6=1000 N/m

FRF Characteristics (Hysteretic Damping)

Again, one can write:

$$([K]+i[D]-\omega^2[M])\{X\}e^{i\omega t}=\{F\}e^{i\omega t}$$

The receptance matrix can be found as:

$$H(\omega)=([K]+i[D]-\omega^2[M])^{-1}=[\Phi][\lambda_r^2-\omega^2][\Phi]^T$$

FRF elements can be extracted:

$$H_{jk}(\omega)=\sum_{r=1}^{N}\frac{\phi_{jr}\phi_{kr}}{\omega_r^2-\omega^2+i\eta_r\omega_r^2}$$

or

$$H_{jk}(\omega)=\sum_{r=1}^{N}\frac{{}_rA_{jk}}{\omega_r^2-\omega^2+i\eta_r\omega_r^2}$$

Modal Constant

MDOF Systems with viscous damping- general case

The general equation of motion for this case can be written as:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\}$$

Consider the zero excitation to determine the natural frequencies and mode shapes of the system:

$$\{x\}=\{X\}e^{st}$$

This leads to:

$$([M]s^2+[C]s+[K])\{X\}=\{0\}$$

This is a complex eigenproblem. In this case, there are 2N eigenvalues but they are in complex conjugate pairs.

MDOF Systems with viscous damping- general case

$$\begin{cases} s_r\,,\ s_r^* \\ \{\psi\}_r\,,\ \{\psi\}_r^* \end{cases} \qquad r = 1,\ldots,N$$

It is customary to express each eigenvalues as:

$$s_r = \omega_r(-\zeta_r + i\sqrt{1-\zeta_r^2})$$

Next, consider the following equation:

$$(s_r^2[M] + s_r[C] + [K])\{\psi\}_r = \{0\}$$

Then, pre-multiply by $\{\psi\}_q^H$:

$$\{\psi\}_q^H(s_r^2[M] + s_r[C] + [K])\{\psi\}_r = \{0\}$$

*

MDOF Systems with viscous damping- general case

A similar expression can be written for $\{\psi\}_q$:

$$(s_q^2[M]+s_q[C]+[K])\{\psi\}_q=\{0\}$$

This can be transposed-conjugated and then multiply by $\{\psi\}_r$

$$\{\psi\}_q^H(s_q^2[M]+s_q[C]+[K])\{\psi\}_r=\{0\} \qquad **$$

Subtract equation * from **, to get:

$$(s_r^2-s_q^2)\{\psi\}_q^H[M]\{\psi\}_r+(s_r-s_q)\{\psi\}_q^H[C]\{\psi\}_r=\{0\}$$

This leads to the first orthogonality equations:

$$(s_r+s_q)\{\psi\}_q^H[M]\{\psi\}_r+\{\psi\}_q^H[C]\{\psi\}_r=\{0\} \qquad (1)$$

MDOF Systems with viscous damping- general case

Next, multiply equation (*) by s_q and (**) by s_r :

$$s_r s_q \{\psi\}_q^H [M]\{\psi\}_r - \{\psi\}_q^H [K]\{\psi\}_r = \{0\} \quad (2)$$

Equations (1) and (2) are the orthogonality conditions:
If we use the fact that the modes are pair, then

$$s_q = \omega_r(-\zeta_r - i\sqrt{1-\zeta_r^2})$$

$$\{\psi\}_q = \{\psi\}_r^*$$

MDOF Systems with viscous damping- general case

Inserting these two into equations (1) and (2):

$$2\omega_r \zeta_r = \frac{\{\psi\}_r^H [C] \{\psi\}_r}{\{\psi\}_r^H [M] \{\psi\}_r} = \frac{c_r}{m_r}$$

$$\omega_r^2 = \frac{\{\psi\}_r^H [K] \{\psi\}_r}{\{\psi\}_r^H [M] \{\psi\}_r} = \frac{k_r}{m_r}$$

Where m_r , k_r , c_r are modal mass, stiffness and damping.

FRF Characteristics (Viscous Damping)

The response solution is:

$$\{X\} = ([K] + i\omega[C] - \omega^2[M])^{-1}\{F\}$$

We are seeking to a similar series expansion similar to the undamped case.

To do this, we define a new vector {u}:

$$\{u\} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}_{2N\times 1}$$

We write the equation of motion as:

$$[C : M]_{N\times 2N}\{\dot{u}\}_{2N\times 1} + [K : 0]\{u\} = \{0\}_{N\times 1}$$

FRF Characteristics (Viscous Damping)

This is N equations and 2N unknowns. We add an identity Equation as:

$$[M:0]\{\dot{u}\}+[0:-M]\{u\}=\{0\}$$

Now, we combine these two equations to get:

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix}\{\dot{u}\}+\begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}\{u\}=\{0\}$$

Which cab be simplified to:

$$[A]\{\dot{u}\}+[B]\{u\}=\{0\}$$

FRF Characteristics (Viscous Damping)

Equation (3) is in a standard eigenvalue form. Assuming a trial solution in the form of $\{u\} = \{U\} e^{st}$

$$(s_r [A] + [B]) \{\theta\}_r = \{0\} \qquad r = 1, \ldots, 2N$$

The orthogonality properties cab be stated as:

$$[\theta]^T [A] [\theta] = [a_r]$$

$$[\theta]^T [B] [\theta] = [b_r]$$

With the usual characteristics:

$$s_r = \frac{b_r}{a_r} \qquad r = 1, \ldots, 2N$$

FRF Characteristics (Viscous Damping)

Let's express the forcing vector as:

$$\{P\}_{2N\times 1} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

Now using the previous series expansion:

$$\begin{Bmatrix} X \\ i\omega X \end{Bmatrix}_{2N\times 1} = \sum_{r=1}^{2N} \frac{\{\theta\}_r^T \{P\} \{\theta\}_r}{a_r (i\omega - s_r)}$$

And because the eigenvalues and vectors occur in complex conjugate pair:

$$\begin{Bmatrix} X \\ i\omega X \end{Bmatrix}_{2N\times 1} = \sum_{r=1}^{N} \frac{\{\theta\}_r^T \{P\} \{\theta\}_r}{a_r (i\omega - s_r)} + \frac{\{\theta\}_r^H \{P\} \{\theta^*\}_r}{a_r^* (i\omega - s_r^*)}$$

FRF Characteristics (Viscous Damping)

Now the receptance frequency response function H_{jk} Resulting from a single force F_k and response parameter X_j

$$H_{jk}(\omega) = \sum_{r=1}^{N} \frac{\theta_{jr}\theta_{kr}}{a_r(\omega_r\zeta_r + i(\omega - \omega_r\sqrt{1-\zeta_r^2})} + \frac{\theta_{jr}^*\theta_{kr}^*}{a_r^*(\omega_r\zeta_r + i(\omega + \omega_r\sqrt{1-\zeta_r^2})}$$

or

$$H_{jk}(\omega) = \sum_{r=1}^{N} \frac{{}_rR_{jk} + i(\omega/\omega_r)({}_rS_{jk})}{\omega_r^2 - \omega^2 + 2i\omega\omega_r\zeta_r}$$

Where:

$$\{{}_rR_k\} = 2(\zeta_r \operatorname{Re}\{{}_rG_k\} - \operatorname{Im}\{{}_rG_k\}\sqrt{1-\zeta_r^2})$$

$$\{{}_rS_k\} = 2\operatorname{Re}\{{}_rG_k\}$$

$$\{{}_rG_k\} = (\theta_{kr}/a_r)\{\theta\}_r$$